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Orthogonal Polynomials in L^1-Approximation

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1. INTRODUCTION

Let U_k, $k \in \mathbb{N}_0$, denote the Chebyshev polynomial of the second kind of degree k. We say that a real function $\tilde{h}$ defined on $[a, b]$, $a, b \in \mathbb{R}$, $a < b$, is a sign function on $[a, b]$ if there is a decomposition of the interval $[a, b]$, $a = x_0 < x_1 < \cdots < x_r = b$, $r \in \mathbb{N}$, such that $\tilde{h}$ or $-\tilde{h}$ takes the value $(-1)^j$ on the interval (x_{j-1}, x_j), $j = 1, ..., r$. $S(\tilde{h}, [a, b])$ denotes the number of changes of sign of $\tilde{h}$ on $[a, b]$.

Many questions on L^1-approximation lead to the following problem:

(a) Let real numbers $b_1, ..., b_n$ be given. Determine a sign function $\tilde{h}$ on $[-1, +1]$ with $S(\tilde{h}, [-1, +1]) = l$ $(\geqslant n)$, such that

$$\int_{-1}^{+1} U_k(x)\, \tilde{h}(x)\, dx = b_{k+1} \qquad \text{for} \quad k = 0, ..., n-1, \tag{1}$$

i.e.,

$$\int_0^{\pi} \sin k\varphi\, h(\varphi)\, d\varphi = b_k \qquad \text{for} \quad k = 1, ..., n,$$

where $h(\varphi) = \tilde{h}(\cos \varphi)$ for $\varphi \in [0, \pi]$.

Under appropriate conditions on the numbers b_k, we describe in this paper all those sign functions which have a finite number of changes of sign and satisfy (1). It is shown that the points at which those sign functions change sign depend in a certain manner on orthogonal polynomials.

For the special case $S(\tilde{h}, [-1, +1]) = n$, problem (a) is of a type similar to the so-called L-problem of moments treated by Ahiezer and Krein [4] and Geronimus [6]. See also [8].

This paper is organised as follows. In Section 2 we solve problem (a) for the special (but very important) case where $S(\tilde{h}, [-1, +1]) = n$. In

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Section 3 we describe all sign functions which solve (1). Section 4 contains applications of the theory to special problems (Posse's problem, L^1-approximation on several intervals, etc.). Finally, we show in Section 5 that there is a close connection between Chebyshev-, L^1-, and L^2-approximation with respect to a suitable weight function on two intervals.

2. Solution of Problem (a) for the Special Case $S(\tilde{h}, [-1, +1]) = n$

In order to state our results we need the following notation. Let D be the open unit disk $\{z \| z| < 1\}$ in the complex plane. As usual we call a function $F: D \to \mathbb{C}$ a Carathéodory function (C-function) if F is analytic in D and $\operatorname{Re} F(z) > 0$ for $z \in D$. It is well known that a function F, normed by $F(0) = 1$, is a C-function if and only if it admits a representation

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \, d\sigma(\varphi) \qquad \text{for} \quad z \in D,$$

where σ is a nondecreasing function with $(1/2\pi) \int_0^{2\pi} d\sigma(\varphi) = 1$. If F is a real C-function, i.e., if F takes real values for real z, then $\sigma(\varphi) = -\sigma(2\pi - \varphi)$ for $\varphi \in [0, 2\pi]$. A bounded nondecreasing function σ on $[0, 2\pi]$ with an infinite set of points of increase will be called a distribution function. Furthermore, we say that a C-function F is nondegenerate if σ is a distribution function, i.e., F is not of the form $ic + \sum_{j=1}^{n} \mu_j((e^{i\varphi_j} + z)/(e^{i\varphi_j} - z))$, where $c \in \mathbb{R}$, $\mu_j \in \mathbb{R}^+$, and $0 \leqslant \varphi_1 < \varphi_2 < \cdots < \varphi_n \leqslant 2\pi$.

If σ is a distribution function on $[0, 2\pi]$ normed by $(1/2\pi) \int_0^{2\pi} d\sigma(\varphi) = 1$, then $P_n(z) = z^n + \cdots$ denotes that polynomial which is orthogonal on the unit circle with respect to the weight $d\sigma$, i.e.,

$$\int_0^{2\pi} e^{-ij\varphi} P_n(e^{i\varphi}) \, d\sigma(\varphi) = 0 \qquad \text{for} \quad j = 0, ..., n-1.$$

$$\Omega_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \{P_n(e^{i\varphi}) - P_n(z)\} \, d\sigma(\varphi) = z^n + \cdots$$

denotes the polynomial of second kind with respect to $d\sigma$.

Let us note that the polynomials P_n resp. Ω_n satisfy a recurrence relation of the type

$$P_{n+1}(z) = zP_n(z) - \bar{a}_n P_n^*(z)$$

resp.

$$\Omega_{n+1}(z) = z\Omega_n(z) + \bar{a}_n \Omega_n^*(z),$$

where $P_n^*(z)=z^n\bar{P}_n(z^{-1})$ denotes the reciprocal polynomial of P_n. The parameters a_n and thus the polynomials P_n resp. Ω_n are real, if $\sigma(\varphi)=-\sigma(2\pi-\varphi)$ for $\varphi\in[0,\pi]$. Polynomials which are orthogonal on the unit circle are studied in detail in [7]; see also [16].

Henceforth we call a sign function $\tilde{h}$ (h) on $[-1,+1]$ ($[0,\pi]$) a normed sign function if $\lim_{\varepsilon\to 0^+}\tilde{h}(1-\varepsilon)=1$ ($\lim_{\varepsilon\to 0^+}h(\varepsilon)=1$).

THEOREM 1. *Let a real sequence $(b_k)_{k\in\mathbb{N}}$ be given and let $F(z)=\exp(-\sum_{k=1}^{\infty}b_kz^k)$. Suppose that F is a nondegenerate C-function with distribution function σ. For each $n\in\mathbb{N}$ let h_n be that normed sign function on $[0,\pi]$ which changes sign exactly at the n zeros of the cosine polynomial* ($z=e^{i\varphi}$, $\varphi\in[0,\pi]$)

$$\frac{\operatorname{Re}\{z^{-(n-1)/2}P_n(z)\}\operatorname{Im}\{z^{-(n-1)/2}\Omega_n(z)\}}{\sin\varphi}.$$

Then

(a) $\int_0^\pi \sin k\varphi\, h_n(\varphi)\,d\varphi=b_k$ *for* $k=1,\ldots,n$,

(b) $\int_0^\pi \sin(n+1)\varphi\, h_n(\varphi)\,d\varphi-b_{n+1}=(4/\pi)\int_0^\pi[\operatorname{Re}\{z^{-(n-1)/2}P_n(z)\}]^2\,d\sigma(\varphi)$,

(c) *there is no other normed sign function g_n with $S(g_n,[0,\pi])\leqslant n$ which satisfies* (a).

Proof. Ad (a). In view of [7, pp. 14–15] and the fact that P_n and Ω_n have real coefficients, the following representations hold:

$$zP_n(z)+P_n^*(z)=(z-1)^{\delta_1}(z+1)^{\delta_2}\prod_{j=1}^{(n+1-\delta_1-\delta_2)/2}(1-2\cos\Psi_jz+z^2)$$

and

$$z\Omega_n(z)-\Omega_n^*(z)=(z-1)^{\delta_1'}(z+1)^{\delta_2'}\prod_{j=1}^{(n+1-\delta_1'-\delta_2')/2}(1-2\cos\varphi_jz+z^2), \tag{2}$$

where

$$\delta_1=\delta_2=0 \quad\text{and}\quad \delta_1'=\delta_2'=1 \quad\text{for } n \text{ odd},$$
$$\delta_1=\delta_2'=0 \quad\text{and}\quad \delta_2=\delta_1'=1 \quad\text{for } n \text{ even},$$

$\Psi_j, \varphi_j \in (0, \pi)$ and $0 < \Psi_1 < \varphi_1 < \Psi_2 < \varphi_2 < \cdots$. Setting $m = (n+1-\delta_1-\delta_2)/2$ and $m' = (n+1-\delta_1'-\delta_2')/2$, we get for $z = e^{i\varphi}$, $\varphi \in [0, \pi]$, that

$$\begin{aligned}
&2\,\mathrm{Re}\{z^{-(n-1)/2} P_n(z)\} \\
&\quad = z^{-(n+1)/2}[zP_n(z) + P_n^*(z)] \\
&\quad = (z-1)^{\delta_1}(z+1)^{\delta_2} z^{-(\delta_1+\delta_2)/2} 2^m \prod_{j=1}^{m} (\cos\varphi - \cos\Psi_j)
\end{aligned}$$

and

$$\begin{aligned}
&2i\,\mathrm{Im}\{z^{-(n-1)/2} \Omega_n(z)\} \\
&\quad = z^{-(n+1)/2}[z\Omega_n(z) - \Omega_n^*(z)] \\
&\quad = (z-1)^{\delta_1'}(z+1)^{\delta_2'} z^{-(\delta_1'+\delta_2')/2} 2^{m'} \prod_{j=1}^{m'} (\cos\varphi - \cos\varphi_j).
\end{aligned}$$

Since (see [7, Theorem 6.1])

$$\begin{aligned}
\frac{\Omega_n^*(z) - z\Omega_n(z)}{P_n^*(z) + zP_n(z)} &= F(z) + O(z^{n+1}) \\
&= \exp\left(-\sum_{k=1}^{n} b_k z^k\right) + O(z^{n+1}) \qquad (3)
\end{aligned}$$

for $z \in D$, we obtain that

$$\begin{aligned}
&\ln\left(-\frac{(z-1)^{\delta_1'}(z+1)^{\delta_2'} \prod_{j=1}^{m'} (1 - 2\cos\varphi_j z + z^2)}{(z-1)^{\delta_1}(z+1)^{\delta_2} \prod_{j=1}^{m} (1 - 2\cos\Psi_j z + z^2)}\right) \\
&\quad = -\sum_{k=1}^{n} b_k z^k + O(z^{n+1}), \qquad (4)
\end{aligned}$$

where the principal branch of ln is chosen. Using the series expansion

$$\ln(1 - 2\cos\varphi z + z^2) = -2 \sum_{k=1}^{\infty} \frac{\cos k\varphi}{k} z^k \qquad (5)$$

we deduce that

$$\begin{aligned}
b_k &= \frac{2}{k}\left\{\sum_{j=1}^{m'} \cos k\varphi_j - \sum_{j=1}^{m} \cos k\Psi_j + \frac{1-(-1)^{k+n}}{2}\right\} \\
&= \int_0^{\pi} \sin k\varphi\, h_n(\varphi)\, d\varphi \qquad \text{for} \quad k = 1, ..., n, \qquad (6)
\end{aligned}$$

where the last equality follows by direct calculation.

(b) From the relations (see [7, (18.11) and (18.12)])

$$F(z)\,P_n^*(z)-\Omega_n^*(z)=\frac{2a_nK_nz^{n+1}}{c_0}+O(z^{n+2})$$

and

$$F(z)\,P_n(z)+\Omega_n(z)=\frac{2K_nz^n}{c_0}+O(z^{n+1})$$

for $n\in\mathbb{N}_0$, where $c_0=(1/2\pi)\int_0^{2\pi}d\sigma(\varphi)=1$ and $K_n=(1/2\pi)\int_0^{2\pi}|P_n(z)|^2\,d\sigma(\varphi)$, it follows that for $n\in\mathbb{N}_0$ and $z\in D$

$$F(z)-\frac{\Omega_n^*(z)-z\Omega_n(z)}{P_n^*(z)+zP_n(z)}=2K_n(1+a_n)\,z^{n+1}+O(z^{n+2}). \tag{7}$$

Furthermore, let us note (see [7, (4.1) and (31.12)]) that

$$K_n(1+a_n)=\frac{K_{n+1}}{(1-a_n)}=\frac{1}{4\pi}\int_0^{2\pi}\left[\frac{P_{n+1}(z)+P_{n+1}^*(z)}{z^{(n+1)/2}(1-a_n)}\right]^2 d\sigma(\varphi).$$

Using the relation

$$P_{n+1}(z)+P_{n+1}^*(z)=(1-a_n)[zP_n(z)+P_n^*(z)]$$

we obtain that

$$K_n(1+a_n)=\frac{1}{\pi}\int_0^{2\pi}[\mathrm{Re}\{z^{-(n-1)/2}P_n(z)\}]^2\,d\sigma(\varphi). \tag{8}$$

From (4), (5), and (6) it follows that for $z\in D$

$$\begin{aligned}\ln &-\frac{(z-1)^{\delta_1'}(z+1)^{\delta_2'}\prod_{j=1}^{m'}(1-2\cos\varphi_jz+z^2)}{(z-1)^{\delta_1}(z+1)^{\delta_2}\prod_{j=1}^{m}(1-2\cos\Psi_jz+z^2)}\\ &=-\left(b_1z+\cdots+b_nz^n+\sum_{k=n+1}^{\infty}b_{k,n}z^k\right),\end{aligned}$$

where

$$b_{k,n}=\int_0^{\pi}\sin k\varphi\,h_n(\varphi)\,d\varphi\qquad\text{for}\quad k\geqslant n+1. \tag{9}$$

Using relation (2) we obtain that

$$\begin{aligned}\frac{\Omega_n^*(z)-z\Omega_n(z)}{P_n^*(z)+zP_n(z)}&=\exp\left(-\sum_{k=1}^{n+1}b_kz^k+(b_{n+1}-b_{n+1,n})\,z^{n+1}\right)+O(z^{n+2})\\ &=[F(z)+O(z^{n+2})][1+(b_{n+1}-b_{n+1,n})\,z^{n+1}+O(z^{2n+2})].\end{aligned}$$

Since $F(0)=1$ we get that

$$F(z)-\frac{\Omega_n^*(z)-z\Omega_n(z)}{P_n^*(z)+zP_n(z)}=(b_{n+1,n}-b_{n+1})\,z^{n+1}+O(z^{n+2}).$$

The assertion follows now from (7), (8), and (9).

(c) Concerning part (c), assume that there is an other normed sign function g_n with $S(g_n, [0,\pi])\leqslant n$ which satisfies (a). Then

$$\int_0^\pi \sin k\varphi[h_n(\varphi)-g_n(\varphi)]\,d\varphi=0 \qquad \text{for} \quad k=1,\ldots,n. \tag{10}$$

Since $S(h_n-g_n, [0,\pi])\leqslant n-1$, there is a sine polynomial $s\neq 0$ of degree $\leqslant n$, such that $\operatorname{sgn} s(\varphi)\operatorname{sgn}[h_n(\varphi)-g_n(\varphi)]\geqslant 0$. Using the fact that $h_n\neq g_n$ on a set of positive measure, it follows that

$$\int_0^\pi s(\varphi)[h_n(\varphi)-g_n(\varphi)]\,d\varphi>0,$$

which is a contradiction to (10).

Remark 1. Let us note (see [7, pp. 2–4 and 6–7]) that the polynomials P_n and Ω_n of Theorem 1 depend on $b_1,\ldots,b_n$ only.

THEOREM 2. *If for each $n\in\mathbb{N}$ there is a normed sign function h_n on $[0,\pi]$ such that $S(h_n,[0,\pi])=n$ and*

$$\int_0^\pi \sin k\varphi\, h_n(\varphi)\,d\varphi=b_k \qquad \textit{for} \quad k=1,\ldots,n,$$

then $F(z)=\exp(-\sum_{k=1}^\infty b_k z^k)$ is a nondegenerate C-function and h_n, $n\in\mathbb{N}$, changes sign exactly at the n zeros of ($z=e^{i\varphi}$, $\varphi\in[0,\pi]$)

$$\frac{\operatorname{Re}\{z^{-(n-1)/2}P_n(z)\}\,\operatorname{Im}\{z^{-(n-1)/2}\Omega_n(z)\}}{\sin\varphi}.$$

Proof. Suppose that h_n changes sign exactly at the n points $\Psi_1,\ldots,\Psi_{[(n+1)/2]},\varphi_1,\ldots,\varphi_{[n/2]}$, where $0<\Psi_1<\varphi_1<\Psi_2<\varphi_2\ldots$. Putting

$$s_{n+1}(z)=(z-1)^{\delta_1'}(z+1)^{\delta_2'}\prod_{j=1}^{m'}(1-2\cos\varphi_j z+z^2)$$

and

$$r_{n+1}(z)=(z-1)^{\delta_1}(z+1)^{\delta_2}\prod_{j=1}^{m}(1-2\cos\Psi_j z+z^2),$$

where δ_1, δ_2, δ_1', δ_2', m, and m' are defined as in the proof of Theorem 1, we obtain with the help of (4), (5), and (6) that

$$-\frac{s_{n+1}(z)}{r_{n+1}(z)}=\exp\left(-\sum_{k=1}^{n} b_k z^k\right)+O(z^{n+1}) \tag{11}$$

for $z \in D$. On the other hand, partial fraction expansion gives, by setting $\Psi_{(n+2)/2}=\pi$ for n even, that

$$-\frac{s_{n+1}(z)}{r_{n+1}(z)}=\sum_{j=1}^{[(n+2)/2]} \lambda_j \frac{1-z^2}{1-2\cos\Psi_j z+z^2}=1+\sum_{k=1}^{\infty} d_k z^k, \tag{12}$$

where, since s_{n+1} and r_{n+1} have interlacing zeros, $\lambda_j \in \mathbb{R}^+$ for $j=1, ..., [(n+2)/2]$ and

$$d_k=2\sum_{j=1}^{[(n+2)/2]} \lambda_j \cos k\Psi_j \qquad \text{for} \quad k \in \mathbb{N}.$$

Since $\lambda_j \in \mathbb{R}^+$ it follows (see, e.g., [4]) that the sequence $\{d_k\}_0^n$, where $d_0=2$, is positive definite on the circumference. Hence $\{d_k\}_0^\infty$ is positive definite on the circumference. By the Herglotz–Riesz theorem (see, e.g., [4, p. 45]), (11), and (12), it follows that F is a nondegenerate C-function. In view of Theorem 1(a) and 1(c) the assertion is proved.

Next let us state some facts about the connection between polynomials which are orthogonal on the unit circle and polynomials which are orthogonal on $[-1, +1]$ (see [7, 16]).

Notation. Let Ψ be a distribution function on $[-1, +1]$ with $\int_{-1}^{+1} d\Psi(x)=1$ and let v be a nonnegative polynomial on $[-1, +1]$. Then p_n^v denotes that polynomial of degree n with leading coefficient one, which is orthogonal to $\mathbb{P}_{n-1}$ on $[-1, +1]$ with respect to the weight $v d\Psi$. Furthermore, let

$$q_{n-1}^v(x)=\int_{-1}^{+1} \frac{v(t)\, p_n^v(t)-v(x)\, p_n^v(x)}{t-x}\, d\Psi(t)$$

denote the polynomial of second kind of vp_n^v with respect to the weight $d\Psi$.

For the following lemma see [7, 16].

LEMMA 1. *Let $\sigma(\varphi)=-\pi\Psi(\cos\varphi)$ for $\varphi \in [0, \pi]$ and $\sigma(\varphi)=\pi\Psi(\cos\varphi)$ for $\varphi \in (\pi, 2\pi]$.*

$$\text{(a)} \quad 2^{-m+1}\,\mathrm{Re}\{z^{-m+1}P_{2m-1}(z)\} = p_m(x),$$

$$2^{-m+1}\frac{\mathrm{Im}\{z^{-m+1}P_{2m-1}(z)\}}{\sin\varphi} = p_{m-1}^{(1-x^2)}(x),$$

$$2^{-m+1}\frac{\mathrm{Im}\{z^{-m+1}\Omega_{2m-1}(z)\}}{\sin\varphi} = q_{m-1}(x).$$

$$\text{(b)} \quad 2^{-m}\frac{\mathrm{Re}\{z^{-m+1/2}P_{2m}(z)\}}{\cos\varphi/2} = p_m^{(1+x)}(x),$$

$$2^{-m}\frac{\mathrm{Im}\{z^{-m+1/2}P_{2m}(z)\}}{\sin\varphi/2} = p_m^{(1-x)}(x),$$

$$2^{-m}\frac{\mathrm{Im}\{z^{-m+1/2}\Omega_{2m}(z)\}}{\sin\varphi/2} = q_{m-1}^{(1+x)}(x)$$

$$= q_m(x) - \frac{p_{m+1}(-1)}{p_m(-1)}\,q_{m-1}(x)$$

for $x = \frac{1}{2}(z + 1/z)$, $z = e^{i\varphi}$, $\varphi \in [0, \pi]$.

$$\text{(c)} \quad \frac{1}{\pi}\int_0^\pi [\mathrm{Re}\{z^{-m+1}P_{2m-1}(z)\}]^2\, d\sigma(\varphi) = 2^{2m-2}\int_{-1}^{+1} p_m^2(x)\, d\Psi(x),$$

$$\frac{1}{\pi}\int_0^\pi [\mathrm{Re}\{z^{-m+1/2}P_{2m}(z)\}]^2\, d\sigma(\varphi)$$

$$= 2^{2m-1}\int_{-1}^{+1} [p_m^{(1+x)}(x)]^2(1+x)\, d\Psi(x).$$

Proof. Parts (a) and (c) can be found in [7, Sect. 30; 16, p. 294]. Part (b) can be proved by the same methods.

Remark 2. If Ψ is absolutely continuous and $\Psi'(x) = w(x)$, then σ is absolutely continuous with $\sigma'(\varphi) = w(\cos\varphi)|\sin\varphi|$.

Remark 3. From Lemma 1 and (2) we obtain the well-known fact that the zeros of p_m and q_{m-1} resp. $p_m^{(1+x)}$ and $q_{m-1}^{(1+x)}$ separate each other, where the greatest zero of $p_m^{(1+x)}$ is greater than the greatest zero of $q_{m-1}^{(1+x)}$. Furthermore, we get from Lemma 1 that the zeros of p_m and $p_{m-1}^{(1-x^2)}$ resp. $p_m^{(1+x)}$ and $p_m^{(1-x)}$ separate each other, where the greatest zero of $p_m^{(1+x)}$ is greater than the greatest zero of $p_m^{(1-x)}$.

LEMMA 2. *Let F with $F(0) = 1$ be a real nondegenerate C-function with distribution function σ and let $\tilde{\sigma}$ denote the distribution function of the real* *nondegenerate C-function* $1/F$. *Let* $\tilde{p}_m$ *be that polynomial which is orthogonal on* $[-1, +1]$ *with respect to the weight* $d\tilde{\Psi}$, *where* $\pi\tilde{\Psi}(\cos\varphi) = -\tilde{\sigma}(\varphi)$ *for* $\varphi \in [0, \pi]$. *Then*

$$\text{(a)} \quad 2^{-m+1}\operatorname{Re}\{z^{-m+1}\Omega_{2m-1}(z)\} = \tilde{p}_m(x)$$

$$2^{-m+1}\frac{\operatorname{Im}\{z^{-m+1}\Omega_{2m-1}(z)\}}{\sin\varphi} = \tilde{p}_{m-1}^{(1-x^2)}$$

$$\text{(b)} \quad 2^{-m}\frac{\operatorname{Re}\{z^{-m+1/2}\Omega_{2m}(z)\}}{\cos\varphi/2} = \tilde{p}_m^{(1+x)}$$

$$2^{-m}\frac{\operatorname{Im}\{z^{-m+1/2}\Omega_{2m}(z)\}}{\sin\varphi/2} = \tilde{p}_m^{(1-x)},$$

where Ω_n *denotes as above the polynomial of second kind with respect to* $d\sigma$.

Proof. Since $\operatorname{Re} F(z) > 0$ implies that $\operatorname{Re}\{1/F(z)\} = 1/|F(z)|^2 \operatorname{Re} F(z) > 0$ for $z \in D$, it follows that $1/F$ is a real nondegenerate C-function, admitting a representation

$$\frac{1}{F(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\tilde{\sigma}(\varphi) \qquad \text{for} \quad z \in D,$$

where $\tilde{\sigma}$ is a distribution function with $(1/2\pi)\int_0^{2\pi} d\tilde{\sigma}(\varphi) = 1$.

According to [7, Theorem 5.1] we have, for $n \in \mathbb{N}_0$, $z \in D$, that

$$\frac{\Omega_n^*(z)}{P_n^*(z)} = F(z) + O(z^{n+1}),$$

from which it follows that

$$\frac{P_n^*(z)}{\Omega_n^*(z)} = \frac{1}{F(z)} + O(z^{n+1}).$$

Using the fact that Ω_n resp. P_n satisfy a recurrence relation of the type

$$\Omega_{n+1}(z) = z\Omega_n(z) - (-a_n)\,\Omega_n^*(z)$$

resp.

$$P_{n+1}(z) = zP_n(z) + (-a_n)\,P_n^*(z),$$

we deduce that Ω_n is orthogonal on the unit circle with respect to the weight $d\tilde{\sigma}$. From Lemma 1 the assertion follows.

3. Description of All Solutions of Problem (a)

LEMMA 3. *Suppose that σ is a distribution function on $[0, 2\pi]$ with $\sigma(\varphi) = -\sigma(2\pi - \varphi)$. Let $l \in \mathbb{N}_0$, $0 \leqslant l \leqslant n-1$, be given and put $c_k = \int_0^{2\pi} e^{-ik\varphi}\, d\sigma(\varphi) / \int_0^{2\pi} d\sigma(\varphi)$ for $k = 0, ..., n-l$. There exist two real polynomials S_{n+1}, R_{n+1} with leading coefficient one, which have $n+1$ simple zeros $e^{i\tilde{\varphi}_j}$ resp. $e^{i\tilde{\Psi}_j}$ with $0 \leqslant \tilde{\varphi}_1 < \tilde{\Psi}_1 < \cdots < \tilde{\varphi}_{n+1} < \tilde{\Psi}_{n+1} < 2\pi$, such that*

$$-\frac{S_{n+1}(z)}{R_{n+1}(z)} = 1 + \sum_{k=1}^{n-l} c_k z^k + O(z^{n+1-l})$$

if and only if there exists a real polynomial $q_l(z) = \prod_{j=1}^{l} (z - z_j)$, $z_j \in D$, such that

$$S_{n+1}(z) = zq_l(z)\Omega_{n-l}(z) - q_l^*(z)\,\Omega_{n-l}^*(z)$$

and

$$R_{n+1}(z) = zq_l(z)\,P_{n-l}(z) + q_l^*(z)\,P_{n-l}^*(z).$$

Proof. *Necessity.* Since S_{n+1} and R_{n+1} are real polynomials they are of the form

$$S_{n+1}(z) = (z-1)^{\delta_1'}(z+1)^{\delta_2'} \prod_{j=1}^{(n+1-\delta_1'-\delta_2')/2} (1 - 2z\cos\varphi_j + z^2)$$

resp. (13)

$$R_{n+1}(z) = (z-1)^{\delta_1}(z+1)^{\delta_2} \prod_{j=1}^{(n+1-\delta_1-\delta_2)/2} (1 - 2z\cos\Psi_j + z^2),$$

where $\delta_1', \delta_2', \delta_1, \delta_2 \in \{0, 1\}$, $\varphi_j, \Psi_j \in (0, \pi)$.

Using the fact that the zeros of S_{n+1} and R_{n+1} interlace, it follows that

$$\delta_1' = \delta_2 = 1 \quad \text{and} \quad \delta_2' = \delta_1 = 0 \quad \text{for } n \text{ even},$$

and (14)

$$\delta_1' = \delta_2' = 1 \quad \text{and} \quad \delta_1 = \delta_2 = 0 \quad \text{for } n \text{ odd}.$$

By partial fraction expansion we obtain (compare the proof of Theorem 2) that the sequence $\{d_k\}_0^n$, defined by $d_0 = 2$ and

$$-\frac{S_{n+1}(z)}{R_{n+1}(z)} = 1 + d_1 z + \cdots + d_n z^n + \cdots,$$

is positive definite on the circumference. Now let $\tilde{P}_n$ be that polynomial which is orthogonal to the sequence $\{d_k\}_0^n$. Then it follows that

$$z\tilde{P}_n(z) + \tilde{P}_n^*(z) = R_{n+1}(z) \quad \text{and} \quad z\tilde{\Omega}_n(z) - \tilde{\Omega}_n^*(z) = S_{n+1}(z). \quad (15)$$

Since $d_k = c_k$ for $k = 1, ..., n-l$, we deduce that $\tilde{P}_n$ can be generated by a recurrence relation of the type

$$\tilde{P}_{k+1}(z) = z\tilde{P}_k(z) - \tilde{a}_k \tilde{P}_k^*(z) \qquad \text{for} \quad k = 0, ..., n-1$$

with $|\tilde{a}_k| < 1$ for $k = 0, ..., n-1$ and $\tilde{a}_k = a_k$ for $k = 0, ..., n-1-l$. Thus we obtain that

$$z\tilde{P}_n(z) + \tilde{P}_n^*(z) = zq_l(z)\, P_{n-l}(z) + q_l^*(z)\, P_{n-l}^*(z)$$

and (16)

$$z\tilde{\Omega}_n(z) - \tilde{\Omega}_n^*(z) = zq_l(z)\, \Omega_{n-l}(z) - q_l^*(z)\, \Omega_{n-l}^*(z),$$

where q_l is generated by the recurrence relation

$$q_{k+1}(z) = zq_k(z) - \tilde{a}_{n-1-k} q_k^*(z) \qquad \text{for} \quad k = 0, ..., l-1,$$

with $q_0(z) = 1$. Hence q_l has all zeros in D and by (15) and (16) the necessity part is proved.

Sufficiency. Put $\tilde{P}_{n+1,l} = zq_l P_{n-l}$ and $\tilde{\Omega}_{n+1,l} = zq_l \Omega_{n-l}$. Since $\tilde{P}_{n+1,l}$ $(\tilde{\Omega}_{n+1,l})$ has all zeros in D, we deduce by considering $\arg \tilde{P}_{n+1,l}(e^{i\varphi})$ $(\arg \tilde{\Omega}_{n+1,l}(e^{i\varphi}))$ that the trigonometric polynomials $\mathrm{Re}\{z^{-(n+1)/2}\tilde{P}_{n+1,l}\}$ and $\mathrm{Im}\{z^{-(n+1)/2}\tilde{P}_{n+1,l}\}$ $(\mathrm{Re}\{z^{-(n+1)/2}\tilde{\Omega}_{n+1,l}\}$ and $\mathrm{Im}\{z^{-(n+1)/2}\tilde{\Omega}_{n+1,l}\})$ have all their zeros in $[0, 2\pi)$ and their zeros interlace.

Using the relation (see [7, p. 7])

$$P_{n-l}\Omega_{n-l}^* + \Omega_{n-l} P_{n-l}^* = \kappa z^{n-l}, \qquad \kappa \in \mathbb{R}^+, \tag{17}$$

we obtain that $(z = e^{i\varphi})$

$$\begin{aligned} &\mathrm{Re}\{z^{-(n+1)/2}\tilde{P}_{n+1,l}\}\, \mathrm{Re}\{z^{-(n+1)/2}\tilde{\Omega}_{n+1,l}\} \\ &\qquad + \mathrm{Im}\{z^{-(n+1)/2}\tilde{P}_{n+1,l}\}\, \mathrm{Im}\{z^{-(n+1)/2}\tilde{\Omega}_{n+1,l}\} \\ &\quad = \mathrm{Re}\{\tilde{P}_{n+1,l}\tilde{\Omega}_{n+1,l}\} = (\kappa/2)|q_l|^2, \end{aligned}$$

from which it follows that the zeros of $2\,\mathrm{Re}\{z^{-(n+1)/2}\tilde{P}_{n+1,l}\} = z^{-(n+1)/2}R_{n+1}$ and $2i\,\mathrm{Im}\{z^{-(n+1)/2}\tilde{\Omega}_{n+1,l}\} = z^{-(n+1)/2}S_{n+1}$ interlace.

With the aid of (17) we get by simple calculation that

$$\frac{q_l^*(z)\, \Omega_{n-l}^*(z) - zq_l(z)\, \Omega_{n-l}(z)}{q_l^*(z)\, P_{n-l}^*(z) + zq_l(z)\, P_{n-l}(z)} - \frac{\Omega_{n-l}^*(z)}{P_{n-l}^*(z)} = O(z^{n+1-l}).$$

Hence

$$-\frac{S_{n+1}(z)}{R_{n+1}(z)} = 1 + \sum_{k=1}^{n-l} c_k z^k + O(z^{n+1-l})$$

and the lemma is proved.

THEOREM 3. *Assume that the given real sequence* $(b_k)_{k\in\mathbb{N}}$ *satisfies the assumption of Theorem* 1. *Let* $n\in\mathbb{N}$, $l\in\mathbb{N}_0$, $n>l$, *and suppose that* h_n *is a normed sign function on* $[0,\pi]$ *with* $S(h_n,[0,\pi])=n$. *Then*

$$\int_0^\pi \sin k\varphi\, h_n(\varphi)\, d\varphi = b_k \qquad \textit{for} \quad k=1,\dots,n-l,$$

if and only if there exists a polynomial $q_l(z)=\prod_{j=1}^l(z-z_j)$, $z_j\in D$, *with real coefficients, such that* h_n *changes sign exactly at the* n *zeros of the cosine polynomial*

$$\frac{\operatorname{Re}\{z^{-(n-1)/2}q_l(z)\,P_{n-l}(z)\}\,\operatorname{Im}\{z^{-(n-1)/2}q_l(z)\,\Omega_{n-l}(z)\}}{\sin\varphi}.$$

Proof. Necessity. Suppose that h_n changes sign exactly at the points $\Psi_1,\dots,\Psi_{[(n+1)/2]}$, $\varphi_1,\dots,\varphi_{[n/2]}$, where $0<\Psi_1<\varphi_1<\Psi_2<\varphi_2<\cdots$, and let S_{n+1} and R_{n+1} be defined as in (13). Then we obtain with the aid of (4), (5), and (6) that

$$-\frac{S_{n+1}(z)}{R_{n+1}(z)}=\exp\left(-\sum_{k=1}^{n-l} b_k z^k\right)+O(z^{n+1-l}).$$

From Lemma 3 it follows that there exists a real polynomial $q_l(z)=\prod_{j=1}^l(z-z_j)$, $z_j\in D$, such that

$$\begin{aligned}2\operatorname{Re}\{z^{-(n-1)/2}&q_l(z)\,P_{n-l}(z)\}\\&=z^{-(n+1)/2}R_{n+1}(z)\\&=(z-1)^{\delta_1}(z+1)^{\delta_2}z^{-(\delta_1+\delta_2)/2}\,2^m\prod_{j=1}^m(\cos\varphi-\cos\Psi_j)\end{aligned}\tag{18}$$

and

$$\begin{aligned}2i\operatorname{Im}\{z^{-(n-1)/2}&q_l(z)\,\Omega_{n-l}(z)\}\\&=z^{-(n+1)/2}S_{n+1}(z)\\&=(z-1)^{\delta_1'}(z+1)^{\delta_2'}z^{-(\delta_1'+\delta_2')/2}\,2^{m'}\prod_{j=1}^{m'}(\cos\varphi-\cos\varphi_j),\end{aligned}\tag{19}$$

where $m=(n+1-\delta_1-\delta_2)/2$ and $m'=(n+1-\delta_1'-\delta_2')/2$.

Sufficiency. Putting

$$z^{-(n+1)/2}R_{n+1}(z)=2\operatorname{Re}\{z^{-(n-1)/2}q_l(z)\,P_{n-l}(z)\}$$

and

$$z^{-(n+1)/2}S_{n+1}(z)=2i\operatorname{Im}\{z^{-(n-1)/2}q_l(z)\,\Omega_{n-l}(z)\}$$

it follows from Lemma 3 that

$$\ln -\frac{S_{n+1}(z)}{R_{n+1}(z)} = -\sum_{k=1}^{n-l} b_k z^k + O(z^{n+1-l})$$

and that S_{n+1} resp. R_{n+1} is of the form (18) resp. (19). Using the relations (5) and (6), the sufficiency part is proved.

As a simple consequence of Theorem 3 we obtain a result of the author which enables one to solve the Solotareff problem (see [12, 13]).

COROLLARY 1. *Let $n \in \mathbb{N}$, $l \in \mathbb{N}_0$, $n > l$ and let h_n be a normed sign function on $[0, \pi]$ with $S(h_n, [0, \pi]) = n$. Then*

$$\int_0^\pi \sin k\varphi \, h_n(\varphi) \, d\varphi = 0 \qquad \text{for} \quad k = 1, ..., n-l,$$

if and only if there exists a real polynomial $q_l(z) = \prod_{j=1}^{l} (z - z_j)$, $z_j \in D$, such that h_n changes sign exactly at the n zeros of the cosine polynomial $\operatorname{Im}\{z^{n+1-2l} q_l^2(z)\}/\sin \varphi$.

Proof. For $b_k = 0$, $k = 1, ..., n-l$, the assumptions of Theorem 3 are fulfilled. Since $P_\nu(z) = \Omega_\nu(z) = z^\nu$ for $\nu \in \mathbb{N}_0$, the assertion follows immediately.

4. APPLICATIONS

In the first part of this section we consider the following problem and give some applications of it:

(a′) Let $\alpha, \beta, \mu \in [-1, +1]$. Describe that normed sign function $\tilde{h}_n$ with $S(\tilde{h}_n, [-1, +1]) = n$, which satisfies

$$\int_{-1}^{+1} U_k(x) \, \tilde{h}_n(x) \, dx = -\mu \int_\alpha^\beta U_k(x) \, dx \qquad \text{for} \quad k = 0, ..., n-1. \quad (20)$$

Remark 4. In the following we need the well-known fact (see, e.g., [11]) that a C-function F with $F(0) = 1$ admits a representation

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \lim_{r \to 1} \operatorname{Re} F(re^{i\varphi}) \, d\varphi \qquad \text{for} \quad z \in D,$$

if $\int_0^\varphi |\operatorname{Re} F(re^{i\theta})| \, d\theta$ is uniformly absolutely continuous for $r < 1$.

LEMMA 4. *For* $\alpha, \beta \in [-1, +1]$, $\alpha \leqslant \beta$, $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$, *with* $\{|\alpha|, |\beta|\} \cap \{2|\lambda|\} \neq \{1\}$ *and* $v_1, v_2 \in \{\frac{1}{2}, -\frac{1}{2}\}$, *let*

$$\begin{aligned}
&\pi w_{\alpha,\beta}^{(\lambda, v_1, v_2)}(x) \\
&\quad= \left|\frac{x-\alpha}{x-\beta}\right|^{\lambda} (1-x)^{v_1}(1+x)^{v_2} \qquad \text{for} \quad x \in (-1, \alpha) \cup (\beta, 1), \\
&\quad= \left|\frac{x-\alpha}{x-\beta}\right|^{\lambda} (1-x)^{v_1}(1+x)^{v_2} \cos(\lambda\pi) \qquad \text{for} \quad x \in (\alpha, \beta).
\end{aligned}$$

By $p_{n,\alpha,\beta}^{(\lambda, v_1, v_2)}$ *we denote that polynomial of degree n with leading coefficient one, which is orthogonal to* $\mathbb{P}_{n-1}$ *on* $[-1, 1]$ *with respect to the weight function* $w_{\alpha,\beta}^{(\lambda, v_1, v_2)}$.

(a) *Let* $\tilde{h}_n$ *be a normed sign function with* $S(\tilde{h}_n, [-1, 1]) = n$. *Then*

$$\int_{-1}^{+1} U_k(x)\, \tilde{h}_n(x)\, dx = -2\lambda \int_{\alpha}^{\beta} U_k(x)\, dx \qquad \text{for} \quad k = 0, ..., n-1,$$

if and only if $\tilde{h}_n$, $n \in \mathbb{N}$, *changes sign exactly at the n zeros of the polynomial*

$$\begin{aligned}
&p_{m,\alpha,\beta}^{(\lambda, -1/2, -1/2)} p_{m-1,\alpha,\beta}^{(-\lambda, 1/2, 1/2)} \qquad \text{for} \quad n = 2m-1, \\
&p_{m,\alpha,\beta}^{(\lambda, -1/2, 1/2)} p_{m,\alpha,\beta}^{(-\lambda, 1/2, -1/2)} \qquad \text{for} \quad n = 2m.
\end{aligned}$$

Proof. Setting $\delta_1 = \arccos \beta$, $\delta_2 = \arccos \alpha$, and $h_n(\varphi) = \tilde{h}_n(\cos \varphi)$ for $\varphi \in (0, \pi)$, it follows immediately that

$$\int_{-1}^{1} U_k(x)\, h_n(x)\, dx = -2\lambda \int_{\alpha}^{\beta} U_k(x)\, dx \qquad \text{for} \quad k = 0, ..., n-1$$

is equivalent to

$$\begin{aligned}
\int_0^{\pi} \sin k\varphi\, h_n(\varphi)\, d\varphi &= -2\lambda \int_{\delta_1}^{\delta_2} \sin k\varphi\, d\varphi \\
&= \frac{2\lambda}{k} (\cos k\delta_2 - \cos k\delta_1) =: b_k
\end{aligned}$$

for $k \in \{1, ..., n\}$. Thus we obtain with the help of (5) that for $z \in D$

$$\begin{aligned}
F(z) := \exp\left(-\sum_{k=1}^{\infty} b_k z^k\right) &= \exp\left\{\lambda \ln\left(\frac{1 - 2\cos\delta_2 z + z^2}{1 - 2\cos\delta_1 z + z^2}\right)\right\} \\
&= \left|\frac{1 - 2\cos\delta_2 z + z^2}{1 - 2\cos\delta_1 z + z^2}\right|^{\lambda} \exp\left\{i\lambda \arg\left(\frac{1 - 2\cos\delta_2 z + z^2}{1 - 2\cos\delta_1 z + z^2}\right)\right\},
\end{aligned}$$

from which we deduce that

$$\begin{aligned} f(\varphi) &:= \lim_{r \to 1^-} \operatorname{Re} F(re^{i\varphi}) = \operatorname{Re} F(e^{i\varphi}) \\ &= \left|\frac{\cos \varphi - \cos \delta_2}{\cos \varphi - \cos \delta_1}\right|^{\lambda} \qquad \text{for} \quad \varphi \in (0, \delta_1) \cup (\delta_2, \pi) \\ &= \left|\frac{\cos \varphi - \cos \delta_2}{\cos \varphi - \cos \delta_1}\right|^{\lambda} \cos(\lambda\pi) \qquad \text{for} \quad \varphi \in (\delta_1, \delta_2). \end{aligned}$$

Furthermore, we obtain that

$$\begin{aligned} g(\varphi) = \operatorname{Re}\{1/F(e^{i\varphi})\} &= 1/f(\varphi) \qquad \text{for} \quad \varphi \in (0, \delta_1) \cup (\delta_2, \pi), \\ &= \cos^2(\lambda\pi)/f(\varphi) \qquad \text{for} \quad \varphi \in (\delta_1, \delta_2). \end{aligned}$$

Now let us suppose that $\lambda \in [0, \frac{1}{2}]$. Using the inequality ($z = re^{i\varphi}$, $r \in (0, 1]$) $|z^2 - 2\cos\delta_1 z + 1|^2 = r^2\{((1+r^2)/r) - 2\cos(\varphi - \delta_1)\}\{((1+r^2)/r) - 2\cos(\varphi + \delta_1)\} \geqslant 4r^2\{1 - \cos(\varphi - \delta_1)\}\{1 - \cos(\varphi + \delta_1)\} = 4r^2\{\cos\varphi - \cos\delta_1\}^2$ and the fact that $1/|\cos\varphi - \cos\delta_1|^{\lambda}$ is integrable on $[0, \pi]$, since $-1 < \min\{\cos\delta_1, 2\lambda\} < 1$, we obtain by Lebesgue's theorem that $\int_0^{\varphi} |\operatorname{Re} F(re^{i\theta})|\, d\theta$ is uniformly absolutely continuous for $r < 1$. Thus, by Remark 4, the distribution function σ of the real C-function F is absolutely continuous on $[0, \pi]$ with

$$\sigma'(\varphi) = \operatorname{Re} F(e^{i\varphi}) \qquad \text{for} \quad \varphi \in (0, \pi). \tag{21}$$

Analogously, one demonstrates that (21) holds also for $\lambda \in [-\frac{1}{2}, 0]$. The assertion follows now from Theorem 1, Lemma 1, and Lemma 2.

For the special case $\alpha = -\beta = -1$ and $\lambda \in (-\frac{1}{2}, \frac{1}{2})$, Lemma 4 was proved by Ahiezer and Krein [4, pp. 98–105] and recently proved again by Young *et al.* [17, 18].

In 1880, Posse (see [8, pp. 266–268]) studied the following problem, now known under his name:

What conditions must the numbers $a, b \in \mathbb{R}$, $1 < a < b$, satisfy such that there exists a polynomial $\tilde{Q}_n = x^n + \cdots$ which satisfies

$$\int_0^1 |\tilde{Q}_n| + (-1)^n \int_a^b \tilde{Q}_n \leqslant \int_0^1 |Q_n| + (-1)^n \int_a^b Q_n \tag{22}$$

for all $Q_n \in \mathbb{P}_n$ with leading coefficient one; when the conditions are fulfilled, find a minimizing polynomial.

Posse solved the above problem with the help of elliptic functions. Tranforming (22) to the interval $[-1, \alpha] \cup [\beta, 1]$, we are able to express the minimizing polynomial in terms of orthogonal polynomials.

LEMMA 5. *Let $\alpha, \beta \in (-1, +1)$ with $\alpha \leqslant \beta$. Suppose that there exists a polynomial $\tilde{Q}_n = x^n + \cdots$ such that*

$$\int_{1}^{\alpha} |\tilde{Q}_n| \underset{(-)}{+} \int_{\beta}^{1} \tilde{Q}_n \leqslant \int_{-1}^{\alpha} |Q_n| \underset{(-)}{+} \int_{\beta}^{1} Q_n$$

for all $Q_n \in \mathbb{P}_n$ with leading coefficient one. Then

(a) $\int_{-1}^{\alpha} x^k \operatorname{sgn} \tilde{Q}_n \underset{(-)}{+} \int_{\beta}^{1} x^k = 0$ *for* $k = 0, ..., n-1$.

(b) $S(\tilde{Q}_n, [-1, \alpha]) \geqslant n-1$.

(c) *If* $S(\tilde{Q}_n, [-1, \alpha]) = n-1$, *then* $\tilde{Q}_n(\alpha - \varepsilon) \underset{(>)}{<} 0$ *for sufficiently small* $\varepsilon \in \mathbb{R}^+$.

Proof. (a) Follows by standard arguments.

(b) Assume that $S(\tilde{Q}_n, [-1, \alpha]) \leqslant n-2$. Construct $\bar{Q} \in \mathbb{P}_{n-1}$, such that

$$\begin{aligned} \operatorname{sgn} \bar{Q} &= \operatorname{sgn} \tilde{Q}_n \quad \text{on} \quad (-1, \alpha), \\ &= \underset{(-)}{+}\, 1 \quad \text{on} \quad (\beta, 1). \end{aligned} \tag{23}$$

Then it follows from (a) that

$$0 = \int_{-1}^{\alpha} |\bar{Q}| + \int_{\beta}^{1} |\bar{Q}|,$$

which is a contradiction.

(c) Suppose that there is a $\delta \in \mathbb{R}^+$ such that $\tilde{Q}_n(x) \underset{(<)}{>} 0$ for $x \in (\alpha - \delta, \alpha)$. Then there is a polynomial $\bar{Q} \in \mathbb{P}_{n-1}$, which satisfies (23). But this is a contradiction.

THEOREM 4. *Let $\alpha, \beta \in (-1, +1)$ with $\alpha \leqslant \beta$ and let $n \in \mathbb{N}$. There exists a polynomial $\tilde{Q}_n = x^n + \cdots$ such that*

$$\int_{-1}^{\alpha} |\tilde{Q}_n| + \int_{\beta}^{1} \tilde{Q}_n \leqslant \int_{-1}^{\alpha} |Q_n| + \int_{\beta}^{1} Q_n$$

for all $Q_n \in \mathbb{P}_n$ with leading coefficient one if and only if $p_{m,\alpha,\beta}^{(-1/2,-1/2,-1/2)}$ ($p_{m,\alpha,\beta}^{(-1/2,-1/2,1/2)}$) has no zero in $(\alpha, 1)$, if $n = 2m-1$ ($n = 2m$).

When the above condition is fulfilled then

$$\begin{aligned} \tilde{Q}_n &= p_{m,\alpha,\beta}^{(-1/2,-1/2,-1/2)} p_{m-1,\alpha,\beta}^{(1/2,1/2,1/2)} \quad \textit{for} \quad n = 2m-1, &(24)\\ &= p_{m,\alpha,\beta}^{(-1/2,-1/2,1/2)} p_{m,\alpha,\beta}^{(1/2,1/2,-1/2)} \quad \textit{for} \quad n = 2m, &(25) \end{aligned}$$

is a minimizing polynomial. $\tilde{Q}_n$ is unique, if $S(\tilde{Q}_n, [-1, \alpha]) = n$.

Proof. *Necessity.* Let the sign function $\tilde{h}_n$ be such that

$$\begin{aligned} \tilde{h}_n &= \operatorname{sgn} \tilde{Q}_n \qquad &\text{on} \quad &[-1, \alpha], \\ &= +1 \qquad &\text{on} \quad &(\alpha, 1]. \end{aligned}$$

Then it follows by Lemma 5 that

$$\int_{-1}^{+1} x^k \tilde{h}_n = \int_{\alpha}^{\beta} x^k \qquad \text{for} \quad k = 0, ..., n-1,$$

and $S(\tilde{h}_n, [-1, +1]) = n$. By Lemma 4 we conclude that $\tilde{h}_n$ is equal a.e. on $[-1, +1]$ to the sign of the polynomial given in (24) resp. (25). Observing that $\tilde{h}_n$ has no change of sign on $(\alpha, 1]$, the assertion follows from Remark 3.

Sufficiency. Follows immediately from the fact that

$$\begin{aligned} \int_{-1}^{\alpha} |Q_n| + \int_{\beta}^{1} Q_n \geqslant \int_{-1}^{\alpha} Q_n \operatorname{sgn} \tilde{Q}_n + \int_{\beta}^{1} Q_n &= \int_{-1}^{\alpha} x^n \operatorname{sgn} \tilde{Q}_n + \int_{\beta}^{1} x^n \\ &= \int_{-1}^{\alpha} |\tilde{Q}_n| + \int_{\beta}^{1} \tilde{Q}_n. \end{aligned}$$

COROLLARY 2. *Let $\alpha, \beta \in (-1, +1)$ with $\alpha \leqslant \beta$ and let $n \in \mathbb{N}$. There exists a polynomial $\tilde{Q}_n = x^n + \cdots$ such that*

$$\int_{-1}^{\alpha} |\tilde{Q}_n| - \int_{\beta}^{1} \tilde{Q}_n \leqslant \int_{-1}^{\alpha} |Q_n| - \int_{\beta}^{1} Q_n$$

for all $Q_n \in \mathbb{P}_n$ with leading coefficient one if and only if α and β satisfy the condition of Theorem 4.

Proof. In view of [8, p. 267] there exists a polynomial $\tilde{Q}_n$ such that

$$\int_{-1}^{\alpha} |\tilde{Q}_n| \underset{(-)}{+} \int_{\beta}^{1} \tilde{Q}_n \leqslant \int_{-1}^{\alpha} |Q_n| \underset{(-)}{+} \int_{\beta}^{1} Q_n \qquad \text{for all} \quad Q_n = x^n + ... \in \mathbb{P}_n$$

if and only if

$$\int_{-1}^{\alpha} |p_{n-1}| \underset{(-)}{+} \int_{\beta}^{1} p_{n-1} \geqslant 0 \qquad \text{for all} \quad p_{n-1} \in \mathbb{P}_{n-1}.$$

Observing that $-p_{n-1} \in \mathbb{P}_{n-1}$, the corollary follows immediately from Theorem 4.

For n odd, the minimal solution of Posse's (transformed) problem can be determined with the help of Lemma 5 and Theorem 3.

Next let us consider problem (a′) for the case where $\{\alpha, \beta\} \cap \{2\lambda\} = \{\pm 1\}$, which was excluded in Lemma 4.

LEMMA 6. *Suppose that $\beta \in (-1, +1)$ and let $\tilde{h}_n$ be a normed sign function on $[-1, +1]$ with $S(\tilde{h}_n, [-1, +1]) = n$. Then*

$$\int_{-1}^{+1} U_k(x)\,\tilde{h}_n(x)\,dx = (-1)^{n+1} \int_{-1}^{\beta} U_k(x)\,dx \qquad \textit{for} \quad k = 0, ..., n-1,$$

if and only if $\tilde{h}_n$ changes sign at the zeros of the polynomial

$$[p_m(x, \beta) - d_m\, p_{m-1}(x, \beta)] \frac{[q_m(x, \beta) - d_m q_{m-1}(x, \beta)]}{(x+1)} \qquad \textit{for} \quad n = 2m-1,$$

and at the zeros of the polynomial

$$p_m(x, -1)\, q_m(x, -1) \qquad \textit{for} \quad n = 2m,$$

where $d_m = q_m(-1, \beta)/q_{m-1}(-1, \beta)$, and

$$p_m(x, t) = [T_m(y(t))\, T_{m+1}(y(x)) - T_{m+1}(y(t))\, T_m(y(x))]/(x-t)$$
$$q_m(x, t) = T_m(y(t))\, U_m(y(x)) - T_{m+1}(y(t))\, U_{m-1}(y(x))$$

with $y(x) = (2x - \beta - 1)/(1 - \beta)$. T_k denotes the Chebyshev polynomial of first kind of degree k.

Proof. *Case* (1): $n = 2m$. Let $\tilde{g}_{n-1}$ be that normed sign function which changes sign exactly at the zeros of

$$U_{2m-1}(y(x)) = T_m(y(x))\, U_{m-1}(y(x)),$$

where $y(x) = (2x - \beta - 1)/(1 - \beta)$. Then it is well known that for $k = 0, ..., n-2$

$$\int_{\beta}^{1} U_k(x)\, \tilde{g}_{n-1}(x)\, dx = 0, \qquad \text{i.e.,} \int_{-1}^{1} U_k(x)\, \tilde{g}_{n-1}(x)\, dx = -\int_{-1}^{\beta} U_k(x)\, dx.$$

By Theorem 1 and Lemma 1 the assertion follows.

Case (2): $n = 2m-1$. Setting $b_k = \int_{\delta_1}^{\pi} \sin k\varphi\, d\varphi = \int_{-1}^{\beta} U_{k-1}(x)\, dx$ for $k \in \mathbb{N}$ it follows that

$$H(z) = \exp\left(-\sum_{k=1}^{\infty} b_k z^k\right) = \exp\left\{\frac{1}{2} \ln\left(\frac{1 - 2\cos\delta_1 z + z^2}{1 + 2z + z^2}\right)\right\}$$

is a nondegenerate C-function which has a simple pole at $z = -1$. Since (see the proof of Lemma 4) $\int_0^\theta |\text{Re } H(re^{i\varphi})| \, d\varphi$ is uniformly absolutely continuous on $[0, \pi-\varepsilon] \cup [\pi+\varepsilon, 2\pi]$, $\varepsilon \in \mathbb{R}^+$, we get that

$$H(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \, d\mu(\varphi),$$

where

$$\begin{aligned} \mu'(\varphi) = \text{Re } H(e^{i\varphi}) &= \left| \frac{\cos\varphi - \cos\delta_1}{\cos\varphi + 1} \right|^{1/2} \qquad &&\text{for} \quad \varphi \in (0, \delta_1) \\ &= 0 &&\text{for} \quad \varphi \in (\delta_1, \pi) \end{aligned}$$

and μ has a jump at $z = -1$ of amount

$$\mu(\pi+0) - \mu(\pi-0) = \pi \lim_{z \to -1} \{H(z)(z+1)\} = \pi\sqrt{2}\sqrt{1+\cos\delta_1}.$$

Furthermore, we obtain that

$$\frac{1}{H(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \frac{1}{\mu'(\varphi)} \, d\varphi,$$

where $1/\mu'(\varphi) := 0$ for $\varphi \in (\delta_1, \pi)$.

Since one can demonstrate that $p_m(x, \beta) - d_m p_{m-1}(x, \beta)$ is orthogonal on $[-1, 1]$ with respect to the distribution function ($x = \cos\varphi$)

$$\Psi(\cos\varphi) = -\mu(\varphi)/\pi \qquad \text{for} \quad \varphi \in [0, \pi]$$

and that $[q_m(x, \beta) - d_m q_{m-1}(x, \beta)]/(x+1)$ is orthogonal on $[-1, 1]$ with respect to the weight function

$$w(\cos\varphi) = \sin\varphi/\mu'(\varphi) \qquad \text{for} \quad \varphi \in [0, \pi],$$

the assertion follows from Theorem 1, Lemma 1, and Lemma 2.

THEOREM 5. (a) *For given $\gamma \in (\frac{1}{2}, \infty)$ and $m \in \mathbb{N}_0$ there exists a number $\beta \in (-1, 1)$, such that β is the smallest zero of the polynomial* $p_{m,-1,\beta}^{(1/2-1/2\gamma,-1/2,-1/2)}$ ($p_{m,-1,\beta}^{(1/2-1/2\gamma,1/2,-1/2)}$).

(b) *Let $\gamma \in (\frac{1}{2}, \infty)$ and $m \in \mathbb{N}_0$ be given and assume that $\beta \in (-1, 1)$. Then*

$$\int_{-1}^{\beta} |p(x)| \, dx \leqslant \gamma \int_{\beta}^{1} |p(x)| \, dx \qquad \textit{for all} \quad p \in \mathbb{P}_{2m-2} \; (\mathbb{P}_{2m-1})$$

and

$$\int_{-1}^{\beta} |p^*(x)|\,dx = \gamma \int_{\beta}^{1} |p^*(x)|\,dx$$

if and only if β is that number which satisfies (a).

(c) *The above inequality holds for $\gamma = \frac{1}{2}$ if and only if $\beta \in (-1, +1)$ is such that*

$$-\frac{2m+1}{2m-1} = \frac{U_m(y(-1)) + U_{m-1}(y(-1))}{U_{m-1}(y(-1)) + U_{m-2}(y(-1))}$$

$$\left(-\frac{m+1}{m} = \frac{T_{m+1}(y(-1))}{T_m(y(-1))}\right),$$

where $y(-1) = -(3+\beta)/(1+\beta)$.

Proof. (b) Let $\gamma \in \mathbb{R}^+$. As in [14, pp. 172–174], one shows that the condition

$$\int_{-1}^{\beta} |p| \leqslant \gamma \int_{\beta}^{1} |p| \qquad \text{for all } p \in \mathbb{P}_{n-1} \qquad \text{and} \qquad \int_{-1}^{\beta} |p^*| = \gamma \int_{\beta}^{1} |p^*| \tag{26}$$

is equivalent to

$$\int_{-1}^{\beta} p \operatorname{sgn} p^* - \gamma \int_{\beta}^{1} p \operatorname{sgn} p^* = 0 \qquad \text{for all} \quad p \in \mathbb{P}_{n-1}$$

and p^* has exactly $(n-1)$ simple zeros in $(\beta, 1)$.

Putting $P = (\beta - x)\,p^*$ we deduce that (26) is equivalent to

$$\int_{-1}^{+1} p \operatorname{sgn} P = (1 - 1/\gamma) \int_{-1}^{\beta} p \operatorname{sgn} P \qquad \text{for all} \quad p \in \mathbb{P}_{n-1}, \tag{27}$$

P has n simple zeros in $(-1, +1)$ and the smallest zero of P is β.

For $\gamma \in (1/2, \infty)$ it follows from Lemma 4 that

$$\begin{aligned} &\text{const. } P \\ &= P_{m,-1,\beta}^{(1/2-1/2\gamma,\,-1/2,\,-1/2)} P_{m-1,-1,\beta}^{(1/2\gamma-1/2,\,1/2,\,1/2)} \qquad &&\text{for} \quad n-1 = 2m-2 \\ &= P_{m,-1,\beta}^{(1/2\gamma-1/2,\,-1/2,\,1/2)} P_{m,-1,\beta}^{(1/2-1/2\gamma,\,1/2,\,-1/2)} \qquad &&\text{for} \quad n-1 = 2m-1 \end{aligned}$$

In view of Remark 3, the assertion is proved.

(a) Follows from (b) and the fact that for given $\gamma \in (0, \infty)$ there exists a $\beta \in (-1, +1)$ and a polynomial $p^* \in \mathbb{P}_{n-1}$, such that (26) is fulfilled.

(c) From (27), Lemma 6, and Remark 3, we deduce that (26) holds for $\gamma = \frac{1}{2}$ if and only if β is the smallest zero of

$$\begin{aligned} p_m(x, \beta) - d_m\, p_{m-1}(x, \beta) \qquad &\text{for} \quad n - 1 = 2m - 2, \\ q_m(x, -1) \qquad &\text{for} \quad n = 2m, \end{aligned}$$

where $p_m(x, \beta)$, $q_m(x, -1)$ and d_m are defined in Lemma 6. With the help of the well-known relations

$$\begin{aligned} &T'_m(x) = m U_{m-1}(x), \qquad U_m(y(\beta)) = U_m(-1) = (-1)^m (m+1), \\ &T_m(y(\beta)) = (-1)^m, \end{aligned}$$

we obtain that

$$\frac{2m+1}{2m-1} = \frac{p_m(\beta, \beta)}{p_{m-1}(\beta, \beta)} = \frac{q_m(-1, \beta)}{q_{m-1}(-1, \beta)},$$

resp.

$$-\frac{m+1}{m} = \frac{T_{m+1}(y(-1))}{T_m(y(-1))}.$$

Finally, let us characterize that polynomial of degree n with leading coefficient one which deviates least from zero in the L^1-norm on several disjoint intervals. L^1-approximation on two intervals was studied in [2, 15]. A criterion for solvability of the L-problem of moments on several intervals has been given in [3] (see also [8, pp. 328–329]).

In the following let $E = [-1, \alpha_1] \cup [\beta_1, \alpha_2] \cup \cdots \cup [\beta_{l-1}, \alpha_l] \cup [\beta_l, 1]$ with $-1 < \alpha_1 < \beta_1 < \cdots < \alpha_l < \beta_l < 1$.

LEMMA 7. *Let $\varepsilon_\nu \in \{-1, 1\}$, $\nu = 1, \dots, l$, be given. By $p_{n,\varepsilon}$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_l)$, we denote that polynomial of degree n with leading coefficient one which is orthogonal on E to $\mathbb{P}_{n-1}$ with respect to the weight function*

$$w_\varepsilon(x) = \frac{1}{\sqrt{1-x^2}} \prod_{\nu=1}^{l} \left(\frac{x - \beta_\nu}{x - \alpha_\nu} \right)^{\varepsilon_\nu/2} \qquad \textit{for} \quad x \in E.$$

Then

$$\int_{-1}^{+1} U_k(x)\, \tilde{h}_n(x)\, dx = \sum_{\nu=1}^{l} \varepsilon_\nu \int_{\alpha_\nu}^{\beta_\nu} U_k(x)\, dx \qquad \textit{for} \quad k = 0, \dots, n-1,$$

if and only if $\tilde{h}_n$ *changes sign at the n zeros of the polynomial*

$$\begin{aligned} &p_{m,\varepsilon}\, p_{m-1,-\varepsilon}^{(1-x^2)} && \textit{for} \quad n=2m-1,\\ &p_{m,\varepsilon}^{(1+x)} p_{m,-\varepsilon}^{(1-x)} && \textit{for} \quad n=2m. \end{aligned}$$

Proof. Let $\delta_\nu = \arccos \beta_\nu$ and $\kappa_\nu = \arccos \alpha_\nu$ for $\nu = 1, ..., l$. Simple calculation gives

$$\sum_{\nu=1}^{l} \varepsilon_\nu \int_{\alpha_\nu}^{\beta_\nu} U_{k-1} = \sum_{\nu=1}^{l} \varepsilon_\nu (\cos k\,\delta_\nu - \cos k\kappa_\nu)/k =: b_k \qquad \text{for} \quad k=1, ..., n.$$

Putting

$$\begin{aligned} F_\varepsilon(z) &= \exp\left\{-\sum_{k=1}^{\infty} b_k z^k\right\} = \exp\left\{\sum_{\nu=1}^{l} \frac{\varepsilon_\nu}{2} \ln\left(\frac{1-2\cos\delta_\nu z + z^2}{1-2\cos\kappa_\nu z+z^2}\right)\right\}\\ &= \prod_{\nu=1}^{l} \left|\frac{1-2\cos\delta_\nu z+z^2}{1-2\cos\kappa_\nu z+z^2}\right|^{\varepsilon_\nu/2}\\ &\quad \cdot \exp\left\{i \sum_{\nu=1}^{l} \frac{\varepsilon_\nu}{2} \arg\left(\frac{1-2\cos\delta_\nu z+z^2}{1-2\cos\kappa_\nu z+z^2}\right)\right\}, \end{aligned}$$

we obtain, since

$$\begin{aligned} \arg\left(\frac{1-2\cos\delta_\nu z+z^2}{1-2\cos\kappa_\nu z+z^2}\right) &= 0 && \text{on } [0,\pi]\backslash(\delta_\nu, \kappa_\nu),\\ &= \pi && \text{on } (\delta_\nu, \kappa_\nu), \end{aligned}$$

that

$$\begin{aligned} \operatorname{Re} F_\varepsilon(e^{i\varphi}) &= \prod_{\nu=1}^{l} \left|\frac{\cos\varphi - \cos\delta_\nu}{\cos\varphi-\cos\kappa_\nu}\right|^{\varepsilon_\nu/2} && \text{for} \quad \varphi \in [0,\pi]\backslash \sum_{\nu=1}^{l} [\delta_\nu, \kappa_\nu],\\ &= 0 && \text{otherwise}, \end{aligned}$$

and

$$\operatorname{Re}\{1/F_\varepsilon(e^{i\varphi})\} = 1/\operatorname{Re} F_\varepsilon(e^{i\varphi}) \qquad \text{for} \quad \varphi \in [0,\pi]\backslash \sum_{\nu=1}^{l} [\delta_\nu, \kappa_\nu].$$

Since $\int_0^\varphi |\operatorname{Re} F_\varepsilon(re^{i\theta})|\, d\theta$ is uniformly absolutely continuous (see the proof of Lemma 4), the assertion follows by Remark 4, Theorem 1, Lemma 1 and Lemma 2.

Let us note that the notation introduced above differs from that used in Lemma 4.

THEOREM 6. *Let Q_n be a polynomial which deviates least from zero on E with respect to the L^1-norm among all polynomials of degree n with leading coefficient one. Then* ($\varepsilon = (\varepsilon_1, ..., \varepsilon_l)$)

$$\int_E |Q_n| = 2 \min_{\substack{\varepsilon_\nu \in \{-1, 1\} \\ \nu \in \{1, ..., l\}}} \left\{ \int_E p_{m,\varepsilon}^2 w_\varepsilon \right\} \qquad \text{for} \quad n = 2m-1,$$

$$= 2 \min_{\substack{\varepsilon_\nu \in \{-1, 1\} \\ \nu \in \{1, ..., l\}}} \left\{ \int_E p_{m,\varepsilon}^{(1+x)}]^2 (1+x) w_\varepsilon \right\} \qquad \text{for} \quad n = 2m.$$

If the minimum is attained for $\tilde{\varepsilon} = (\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_l)$ then

$$\tilde{Q}_n = p_{m,\tilde{\varepsilon}} \cdot p_{m-1,-\tilde{\varepsilon}}^{(1-x^2)} \qquad \text{for} \quad n = 2m-1$$

$$= p_{m,\tilde{\varepsilon}}^{(1+x)} \cdot p_{m,-\tilde{\varepsilon}}^{(1-x)} \qquad \text{for} \quad n = 2m$$

is a minimizing polynomial.

Proof. By standard arguments, one shows that Q_n is an L^1-extremal polynomial on E if and only if

$$\int_E U_k \operatorname{sgn} Q_n = 0 \qquad \text{for} \quad k = 0, ..., n-1. \tag{28}$$

Because of (28) it follows that Q_n has n simple zeros in $(-1, +1)$, from which we deduce that there is always a minimizing polynomial $\tilde{Q}_n$ on E, which has n simple zeros in E.

For given, but arbitrary, $\varepsilon_\nu \in \{-1, 1\}$, $\nu = 1, ..., l$, let $S_{2m-1,\varepsilon}$, $\varepsilon = (\varepsilon_1, ..., \varepsilon_l)$, be that polynomial of degree $2m-1$ with leading coefficient one, which satisfies

$$\int_{-1}^{+1} U_k \operatorname{sgn} S_{2m-1,\varepsilon} = \sum_{\nu=1}^{l} \varepsilon_\nu \int_{\alpha_\nu}^{\beta_\nu} U_k \qquad \text{for} \quad k = 0, ..., 2m-2. \tag{29}$$

By Lemma 7, Theorem 1, and Lemma 1 it follows that

$$\int_{-1}^{+1} U_{2m-1} \operatorname{sgn} S_{2m-1,\varepsilon} = \sum_{\nu=1}^{l} \varepsilon_\nu \int_{\alpha_\nu}^{\beta_\nu} U_{2m-1} + 2^{2m} \int_E p_{m,\varepsilon}^2 w_\varepsilon.$$

Since the minimizing polynomial $\tilde{Q}_{2m-1}$ has all zeros in E, we obtain from (28) and (29), setting $\tilde{\varepsilon}_\nu = \operatorname{sgn} \tilde{Q}_{2m-1}(x)$ for $x \in (\alpha_\nu, \beta_\nu)$, that $\tilde{Q}_{2m-1} = S_{2m-1,\tilde{\varepsilon}}$. Using (28) and (29), we find

$$2\int_E p_{m,\tilde{\varepsilon}}^2 w_{\tilde{\varepsilon}} = \int_E |\tilde{Q}_{2m-1}| \leqslant \int_E |S_{2m-1,\varepsilon}|$$

$$= \int_{[-1,1]} |S_{2m-1,\varepsilon}| - \int_{[-1,1]\setminus E} |S_{2m-1,\varepsilon}|$$

$$= \sum_{\nu=1}^{l} \varepsilon_\nu \int_{\alpha_\nu}^{\beta_\nu} S_{2m-1,\varepsilon} + 2\int_E p_{m,\varepsilon}^2 w_\varepsilon - \int_{[-1,1]\setminus E} |S_{2m-1,\varepsilon}|$$

$$\leqslant 2\int_E p_{m,\varepsilon}^2 w_\varepsilon .$$

Thus the first part of the theorem is proved for n odd. The case where n is even is demonstrated in an analogous way.

Furthermore, it follows from Lemma 7 that $\tilde{Q}_n$ is of the given form.

5. Chebyshev Polynomials on Two Disjoint Intervals

Notation. Let $\alpha, \beta \in (-1, +1)$ with $\alpha < \beta$. For abbreviations let $p_n = p_{n,\alpha,\beta}^{(1/2,-1/2,-1/2)}$ resp. $\tilde{p}_n = p_{n,\alpha,\beta}^{(-1/2,-1/2,-1/2)}$ and let $w = w_{\alpha,\beta}^{(1/2,-1/2,-1/2)}$ resp. $\tilde{w} = w_{\alpha,\beta}^{(-1/2,-1/2,-1/2)}$, where $p_{n,\alpha,\beta}^{(\cdot,\cdot,\cdot)}$ and $w_{\alpha,\beta}^{(\cdot,\cdot,\cdot)}$ are defined in Lemma 4. q_{n-1} resp. $\tilde{q}_{n-1}$ denotes the polynomial of second kind of p_n resp. $\tilde{p}_n$. Let us note (see the proof of Lemma 4) that $q_{n-1} = \tilde{p}_n^{(1-x^2)}$ and $\tilde{q}_{n-1} = p_{n-1}^{(1-x^2)}$.

The orthogonal polynomials $\{p_n\}_{n\in\mathbb{N}_0}$ resp. $\{\tilde{p}_n\}_{n\in\mathbb{N}_0}$ satisfy a recurrence relation of the type

$$p_n(x) = (x-\alpha_n)\, p_{n-1}(x) - \lambda_n p_{n-2}(x),$$

resp.

$$\tilde{p}_n(x) = (x-\tilde{\alpha}_n)\, p_{n-1}(x) - \tilde{\lambda}_n p_{n-2}(x).$$

For the recursion coefficients α_n, λ_n resp. $\tilde{\alpha}_n$, $\tilde{\lambda}_n$, a recurrence relation is known (see [15]).

Let us recall (see Theorem 6 or [15]) that the L^1-minimizing polynomial on $[-1, \alpha] \cup [\beta, 1]$ can be constructed with the help of the polynomials p_n and $\tilde{p}_n$. In this section we will show that these polynomials also play an important role in Chebyshev approximation on $[-1, \alpha] \cup [\beta, 1]$.

Notation. We say that a polynomial $\mathcal{T}_n$ is a Chebyshev polynomial (T-polynomial) on $[-1, \alpha] \cup [\beta, 1]$ if $\mathcal{T}_n$ deviates least from zero on $[-1, \alpha] \cup [\beta, 1]$ in the maximum norm among all polynomials of degree n and leading coefficient one.

A description of T-polynomials on two intervals in terms of elliptic functions has been given in [1].

LEMMA 8. *Let $n \in \mathbb{N}$. Then*

(a) $(x-\alpha)\,p_n^2(x)+(x-\beta)(1-x^2)\,q_{n-1}^2(x)=A_n(x+\alpha_{n+1}-(\beta+\alpha)/2)$, *resp.* $(x-\beta)\,\tilde{p}_n^2(x)+(x-\alpha)(1-x^2)\,\tilde{q}_{n-1}^2(x)=\tilde{A}_n(x+\tilde{\alpha}_{n+1}-(\beta+\alpha)/2)$, *where* $A_n=2\int_{-1}^{+1} p_n^2 w$ *resp.* $\tilde{A}_n=2\int_{-1}^{+1}\tilde{p}_n^2\tilde{w}$.

(b) $(x-\alpha)\,p_n(x)\,\tilde{q}_{n-1}(x)-(x-\beta)\,\tilde{p}_n(x)\,q_{n-1}(x)=(A_n-\tilde{A}_n)/2$.

Proof. Part (a) has been given in [15, Lemma 3].

(b) Since (see, e.g., [5, Theorem 1.17])

$$\sqrt{\frac{1-\alpha x}{1-\beta x}}\cdot\frac{1}{\sqrt{1-x^2}}-\left(\int_{-1}^{+1} p_n^2 w\right)x^{2n}+O(x^{2n+1})=\frac{q_{n-1}^*(x)}{p_n^*(x)}$$

and

$$\sqrt{\frac{1-\beta x}{1-\alpha x}}\,\frac{1}{\sqrt{1-x^2}}-\left(\int_{-1}^{+1}\tilde{p}_n^2\tilde{w}\right)x^{2n}+O(x^{2n+1})=\frac{\tilde{q}_{n-1}^*(x)}{\tilde{p}_n^*(x)}$$

we obtain that

$$\frac{(1-\alpha x)\,\tilde{q}_{n-1}^*(x)}{(1-\beta x)\,\tilde{p}_n^*(x)}-\frac{q_{n-1}^*(x)}{p_n^*(x)}=\left(\int_{-1}^{+1} p_n^2 w-\int_{-1}^{+1}\tilde{p}_n^2\tilde{w}\right)x^{2n}+O(x^{2n+1}),$$

from which (b) follows.

LEMMA 9. *The following properties are equivalent*: (1) $\alpha_{n+1}=(\beta-\alpha)/2$; (2) $q_{n-1}(\alpha)=0$; (3) $p_n=\tilde{p}_n$; (4) $\tilde{q}_{n-1}(\beta)=0$; (5) $\tilde{\alpha}_{n+1}=(\alpha-\beta)/2$.

Proof. (1) $\Leftrightarrow$ (2) follows immediately from Lemma 8(a).

(2) $\Rightarrow$ (3): $q_{n-1}(\alpha)=0$ implies by Lemma 8(b) that

$$(x-\alpha)\,p_n\tilde{q}_{n-1}=(x-\beta)\,\tilde{p}_n q_{n-1}. \tag{30}$$

Since $\alpha_{n+1}=(\beta-\alpha)/2$ it follows with the help of Lemma 8(a) that $p_n(\beta)\neq 0$. Using additionally the fact that the zeros of p_n and q_{n-1} strictly interlace, the implication follows from (30).

(3) $\Rightarrow$ (2): $p_n=\tilde{p}_n$ implies by Lemma 8(b) that

$$p_n[(x-\alpha)\,\tilde{q}_{n-1}-(x-\beta)\,q_{n-1}]=A_n-\tilde{A}_n.$$

Hence $(x-\alpha)\,\tilde{q}_{n-1}=(x-\beta)\,q_{n-1}$.

The remaining equivalences are established analogously.

THEOREM 7. *Let $n \in \mathbb{N}$. The following properties are equivalent*:

(a) *$\mathcal{T}_n$ is a T-polynomial on $[-1,\alpha]\cup[\beta,1]$ with $(n+2)$ deviation points.*

(b) $\int_{[-1,\alpha]\cup[\beta,1]} x^k \mathcal{T}_n(x)\, u(x)\, dx = 0$ *for* $k = 0, ..., n$, *where*

$$u(x) = \frac{-1}{\pi\sqrt{(1-x^2)(x-\alpha)(x-\beta)}} \quad \textit{for} \quad x \in (-1, \alpha),$$
$$= \frac{1}{\pi\sqrt{(1-x^2)(x-\alpha)(x-\beta)}} \quad \textit{for} \quad x \in (\beta, 1).$$

(c) $\mathcal{T}_n = p_n = \tilde{p}_n$ ($\mathcal{T}_n$ *attains its maximum at the zeros of* $(x^2-1)(x-\beta)\, q_{n-1}(x)$).

Proof. (a)$\Rightarrow$(b): Since $\mathcal{T}_n$ has $(n+2)$ deviation points it follows from [1, Theorem 11] that $\mathcal{T}_n$ attains its maximum at the boundary points -1, α, β, 1 and at $(n-2)$ points $y_j \in (-1, \alpha) \cup (\beta, 1)$. Setting

$$S_{n-2}(x) = \prod_{j=1}^{n-2} (x - y_j) \tag{31}$$

we get

$$\mathcal{T}_n^2(x) + (1-x^2)(x-\alpha)(x-\beta)\, S_{n-2}^2(x) = L^2, \tag{32}$$

where L is the minimum deviation. Hence we obtain for $x > 1$ that

$$\frac{S_{n-2}(x)}{\mathcal{T}_n(x)} = \frac{1}{\sqrt{(x-\alpha)(x-\beta)(x^2-1)}} + O\left(\frac{1}{x^{2n+2}}\right). \tag{33}$$

With the aid of [10, Theorem 4.1 and pp. 494–495] we get that

$$\frac{1}{\sqrt{(z-\alpha)(z-\beta)(z^2-1)}} = \int_{[-1,\alpha]\cup[\beta,1]} \frac{u(t)}{z-t}\, dt \qquad \text{for} \quad z \in \mathbb{C}\backslash[-1, +1],$$

from which (b) follows.

(b)$\Rightarrow$(c): In view of (33) it follows that $(x > 1)$

$$\frac{(x-\alpha)\, S_{n-2}(x)}{\mathcal{T}_n(x)} = \frac{\sqrt{x-\alpha}}{\sqrt{(x-\beta)(x^2-1)}} + O\left(\frac{1}{x^{2n+1}}\right)$$
$$= \int_{-1}^{+1} \frac{w(t)}{x-t}\, dt + O\left(\frac{1}{x^{2n+1}}\right),$$

which implies that

$$\mathcal{T}_n(x) = p_n(x) \qquad \text{and} \qquad (x-\alpha)\, S_{n-2}(x) = q_{n-1}(x). \tag{34}$$

Analogously one demonstrates $\mathcal{T}_n = \tilde{p}_n$.

(c) ⇒ (a): In view of Lemma 9 and Lemma 8(a) we obtain that

$$(x-\alpha)\, p_n^2(x) + (x-\beta)(1-x^2)\, q_{n-1}^2(x) = A_n(x-\alpha).$$

Setting $S_{n-2}(x) = q_{n-1}(x)/(x-\alpha)$ it follows that

$$p_n^2(x) + (x-\alpha)(x-\beta)(1-x^2)\, S_{n-2}^2(x) = A_n,$$

from which we deduce that p_n is a T-polynomial on $[-1, \alpha] \cup [\beta, 1]$ with $(n+2)$ deviation points and minimum deviation $\sqrt{A_n}$.

COROLLARY 3. *Suppose that $\mathscr{T}_n$ is a T-polynomial on $[-1, \alpha] \cup [\beta, 1]$ with $(n+2)$ deviation points. Then $\mathscr{T}_n \cdot \mathscr{T}'_n/n$ is an L^1-minimizing polynomial on $[-1, \alpha] \cup [\beta, 1]$.*

Proof. Since $\mathscr{T}_n = p_n = \tilde{p}_n$ and by (30), $(x-\beta)\, p_{n-1}^{(1-x^2)} = (x-\alpha)\, \tilde{q}_{n-1} = (x-\alpha)\, p_{n-1}^{(1-x^2)}$, we get from Theorem 6 and (28) that

$$\int_{[-1,\alpha] \cup [\beta, 1]} x^k \operatorname{sgn}(p_n q_{n-1}) = 0 \qquad \text{for} \quad k = 0, ..., 2n-2. \tag{35}$$

Now let S_{n-2} be defined as in (31). Then it follows from (32) that there is a $c \in (\alpha, \beta)$ such that $n(x-c)\, S_{n-2}(x) = \mathscr{T}'_n(x)$. Thus we get by (34) that

$$\operatorname{sgn}(p_n q_{n-1}) = \operatorname{sgn}(\mathscr{T}_n \cdot \mathscr{T}'_n) \qquad \text{on } (-1, \alpha) \cup (\beta, 1).$$

Because of (35) the assertion is proved.

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